Topology in Physics 2018 - lecture 12

Marcel Vonk

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We are on our way towards understanding the "physics proof" of the index theorem. Today, our goal will be to rewrite the index of an elliptic Fredholm operator D twice:

1. We want to write the index as the trace of a **heat kernel**,

2. Then, we want to rewrite this trace in terms of a **path integral** of a supersymmetric action.

By the end of this lecture, we will have motivated this path integral formula. The final part of the proof, in one of the next lectures, will then be to evaluate the path integral and obtain the result that Atiyah and Singer found.

Again, we follow closely the discussion in Nakahara, which can be mostly found in sections 1.2, 1.5, 12.7, 12.9 and 12.10.

12.1 The heat kernel

The heat kernel was already briefly introduced in the previous lecture. Here, we want to give a recap and rephrase some statements that were made in that lecture in terms of quantum mechanics. Let $\Delta : \Gamma(M, E) \to \Gamma(M, E)$ be an elliptic Fredholm operator, which we now think of as "acting on wave functions". We assume Δ has a countable set of eigenstates $|n\rangle$ with nonnegative eigenvalues λ_n:

$$\Delta|n\rangle = \lambda_n|n\rangle, \qquad \lambda_n \geq 0. \tag{12.1}$$

Moreover, we will assume that the $|n\rangle$ form a complete basis of states. The typical example to have in mind for such a Δ is the Laplacian on a compact manifold.

The fact that Δ is Fredholm means that it has a finite number of eigenstates with eigenvalue 0. Let us introduce an additional notation for those particular states: we label them as

$$|0, i\rangle \qquad \text{with} \qquad 1 \leq i \leq n_0 = \dim \ker \Delta. \tag{12.2}$$

Now the **heat kernel** for Δ is the operator

$$h(t) = e^{-t\Delta}. \tag{12.3}$$

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The name "heat kernel" for this operator will be clarified in exercise 1. To be able to compute with it, it is useful to represent it in the basis of Δ-eigenstates. That is, we insert this complete set of states to express the heat kernel as

$$\begin{aligned} h(t) &= \sum_n e^{-t\Delta}|n\rangle\langle n| \\ &= \sum_n e^{-t\lambda_n}|n\rangle\langle n| \end{aligned} \qquad (12.4)$$

In particular, we can compute the trace of the heat kernel[1]:

$$\begin{aligned} \text{Tr } h(t) &= \sum_m \langle m|h(t)|m\rangle \\ &= \sum_n e^{-t\lambda_n} \end{aligned} \qquad (12.5)$$

The reason the heat kernel is interesting for us, is that in the limit $t \to \infty$, only the eigenstates with $\lambda_n = 0$ contribute, and so in that limit the heat kernel reduces to a projection operator on those states:

$$\lim_{t\to\infty} h(t) = \sum_i |i,0\rangle\langle i,0|. \qquad (12.6)$$

and

$$\lim_{t\to\infty} \text{Tr } h(t) = \dim \ker \Delta. \qquad (12.7)$$

Now, let us study the particular case where Δ is of the form

$$\Delta_E = D^\dagger D \qquad (12.8)$$

with $D : \Gamma(M, E) \to \Gamma(M, F)$, and let us also consider the "backward" operator

$$\Delta_F = DD^\dagger. \qquad (12.9)$$

We will be interested in the difference of the traces of the respective heat kernels of those operators:

$$I_{E,F} = \text{Tr } h_E(t) - \text{Tr } h_F(t). \qquad (12.10)$$

It appears that $I_{E,F}$ depends on t, but the interesting result is that in fact it does not! The reason for this is that for *nonzero* eigenvalues λ_n, there is a one-to-one mapping between eigenstates of Δ_E and eigenstates of Δ_F. In fact, let $|n\rangle$ be such an eigenstate for E. Then $D|n\rangle$ is nonzero since $D^\dagger D|n\rangle$ is nonzero. Moreover, $D|n\rangle$ is an eigenstate of Δ_F, with the same eigenvalue:

$$\Delta_F(D|n\rangle) = DD^\dagger D|n\rangle = D\lambda_n|n\rangle = \lambda_n(D|n\rangle). \qquad (12.11)$$

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[1] Here, we assume the sum converges, which is often the case at least for large enough t, but more generally this may require regularization techniques.

Vice versa, $D^\dagger$ maps eigenstates of Δ_F to those of Δ_E, and up to a factor λ_n^2 is in fact the inverse map. Thus, we see that

$$
\begin{aligned}
I_{E,F} &= \operatorname{Tr} h_E(t) - \operatorname{Tr} h_F(t) \\
&= \sum_{n_E} e^{-t\lambda_{n_E}} - \sum_{n_F} e^{-t\lambda_{n_F}} \\
&= \sum_{i_E} 1 - \sum_{i_F} 1 \\
&= \dim \ker \Delta_E - \dim \ker \Delta_F
\end{aligned}
\tag{12.12}
$$

where in the third line we sum only over states with eigenvalue zero, as all other contributions to the second line cancel one by one. Thus, our result is indeed t-independent. Moreover, it looks a lot like an index, but it is not quite, since Δ_F is *not* the adjoint of Δ_E. (In fact, both operators are self-adjoint and have index zero.) Let us now see how nonetheless, this result can help us to compute actual indices.

12.2 A path integral formula for the index

We want to use what we have learned about heat kernels to rewrite our formula for the index of an elliptic Fredholm operator D. We start from the usual expression:

$$
\operatorname{ind}(D) = \dim \ker D - \dim \ker D^\dagger
\tag{12.13}
$$

Now, we note that $f \in \ker D$ of course implies that $f \in \ker D^\dagger D$. However, the converse is also true:

$$
D^\dagger D f = 0 \quad \Rightarrow \quad \langle f | D^\dagger D f \rangle = 0 \quad \Rightarrow \quad \langle D f | D f \rangle = 0 \quad \Rightarrow \quad D f = 0.
\tag{12.14}
$$

Thus, we see that

$$
\ker D = \ker D^\dagger D
\tag{12.15}
$$

and a similar argument shows that

$$
\ker D^\dagger = \ker D D^\dagger.
\tag{12.16}
$$

As a result, we can rewrite our index formula as

$$
\operatorname{ind}(D) = \dim \ker D^\dagger D - \dim \ker D D^\dagger.
\tag{12.17}
$$

This is exactly the expression that we found in (12.12)! Thus, we can now reverse the logic and write

$$
\operatorname{ind}(D) = \operatorname{Tr} e^{-\beta D^\dagger D} - \operatorname{Tr} e^{-\beta D D^\dagger},
\tag{12.18}
$$

where in particular the answer does not depend on the choice of the real positive number β. The change of notation from t to β is purely conventional. (In quantum mechanics, if Δ is a Hamiltonian, one often uses t for Lorentzian time and β for Euclidean time.)

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Recall that D maps sections of a bundle E to sections of a bundle F, and $D^\dagger$ maps in the oposite direction. To be able to combine the actions of D and $D^\dagger$, we now introduce the bundle $G = E \oplus F$ and introduce a new operator $Q : \Gamma(G, M) \to \Gamma(G, M)$ as

$$iQ = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}. \tag{12.19}$$

The physical interpretation of this construction is that we want to think of sections of E as "bosons", sections of F as "fermions", and of Q as a "supersymmetry operator" mapping bosons to fermions and vice versa. The factor of i is conventional; it is such that analogously to supersymmetric systems, one can introduce the "Hamiltonian"

$$H = -Q^2 = \begin{pmatrix} D^\dagger D & 0 \\ 0 & DD^\dagger \end{pmatrix}. \tag{12.20}$$

We can now *almost* write the index as the partition function $\mathrm{Tr}\ e^{-\beta H}$ of this Hamiltonian, except for the minus sign in (12.18). To solve this discrepancy, we introduce the "fermion counting operator"

$$(-1)^F \equiv \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{12.21}$$

Using this operator, we can write the index as a "twisted partition function",

$$\mathrm{ind}(D) = \mathrm{Tr}\ (-1)^F e^{-\beta H} \tag{12.22}$$

So far, we have simply been rewriting the index formula, but of course the above formula has a very "physical smell" to it. It turns out we can indeed use physics techniques to compute the right hand side. In fact, our next step will be to rewrite this expression as a path integral:

$$\mathrm{ind}(D) = \int_{PBC} Dx D\psi\ e^{-\int_0^\beta L dt} \tag{12.23}$$

where $L[x, \psi]$ is a Lagrangian to be introduced later. Note that it may be expected that such an expression exists: we have already seen in lecture 2 that expressions of the form

$$\langle \phi | e^{-tH} | \psi \rangle \tag{12.24}$$

can be interpreted as propagating $|\psi\rangle$ in time and computing the inner product (overlap) of the resulting state with $|\phi\rangle$. In that case, we have seen how the result can be written in terms of a path integral with initial field configuration corresponding to $\psi(x)$ and final field configuration $\phi(x)$.

Here, we want to do something similar, but there are two differences:

1. There is a **trace** in the expression (12.22).

2. There is the insertion of the **fermion number** operator $(-1)^F$.

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As it turns out, these two particularities work together nicely: together, they imply that in the path integral we should not integrate with *fixed* "in" and "out" boundary conditions, but we should only require that these two boundary conditions are *equal*. That is, we want to integrate using **periodic boundary conditions** (hence the "PBC" in formula (12.23))

$$x(t = 0) = x(t = \beta) \qquad \psi(t = 0) = \psi(t = \beta). \tag{12.25}$$

For the bosonic fields x, this follows from particularity (1) only: the presence of a trace. In fact, taking the trace means exactly that: sum over all "in" states with exactly the same "out" state. However, for fermions, there is a subtlety and we actually need particularity (2) to arrive at periodic boundary conditions. We will make this precise in the next section; then in the section after that we will come to the question what the Lagrangian L is.

12.3   The fermionic harmonic oscillator: path integral

The fact that a trace *and* an insertion of $(-1)^F$ for fermions leads to periodic boundary conditions is explained most easily in an example: that of the **fermionic harmonic oscillator**. We will study this case in detail, and then state without a detailed proof that exactly the same result holds more generally.

To introduce the fermionic harmonic oscillator, let us first recall some basics for the ordinary, *bosonic* harmonic oscillator. This quantum mechanical system has a Hamiltonian with a quadratic potential:

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 \tag{12.26}$$

Here, for simplicity, we have chosen units where the mass and the frequency of the oscillator are equal to 1. Moreover, we will set $\hbar = 1$ so that the commutator for the operators p and x is

$$[x, p] = i. \tag{12.27}$$

To solve the quantum problem (that is, to find the eigenstates and eigenvalues of H), it is very useful to introduce *annihilation* and *creation* operators:

$$a = \frac{1}{\sqrt{2}}(x + ip) \qquad a^\dagger = \frac{1}{\sqrt{2}}(x - ip) \tag{12.28}$$

satisfying

$$[a, a^\dagger] = 1. \tag{12.29}$$

One can then define the ground state $|0\rangle$ to be the (normalized) state for which

$$a|0\rangle = 0 \tag{12.30}$$

and higher excited states inductively as

$$|n\rangle = \frac{1}{\sqrt{n}}a^\dagger|n - 1\rangle, \tag{12.31}$$

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where the prefactor of $1/\sqrt{n}$ makes sure that the state is properly normalized. If you are not that familiar with the above construction, it is instructive to show that using the x-basis, where p is the operator $-i\partial_x$, this reproduces the usual energy eigenfunctions of the harmonic oscillator.

To find the corresponding eigenvalues, one introduces the number operator

$$N = a^\dagger a \tag{12.32}$$

One can easily show that $N|n\rangle = n|n\rangle$ and that

$$H = \frac{1}{2}\left(a^\dagger a + aa^\dagger\right) = N + \frac{1}{2} \tag{12.33}$$

so that the energy E_n of the state $|n\rangle$ is

$$E_n = n + \frac{1}{2} \tag{12.34}$$

This reproduces (recall that we set $\omega = \hbar = 1$) exactly the well-known spectrum of the harmonic oscillator.

Our aim is now to generalize the above to a fermionic setting. To this end, we want to introduce creation and annihilation operators that satisfy *anti*-commutation relations:

$$\{c, c^\dagger\} = cc^\dagger + c^\dagger c = 1, \qquad \{c, c\} = \{c^\dagger, c^\dagger\} = 0. \tag{12.35}$$

In exercise 3, we will see that the last two relations in fact follow naturally once we impose the first.

In the Hamiltonian, we similarly introduce a minus sign:

$$H = \frac{1}{2}\left(c^\dagger c - cc^\dagger\right) = N - \frac{1}{2} \tag{12.36}$$

where as in the bosonic case we introduced the number operator

$$N = c^\dagger c. \tag{12.37}$$

What are the eigenstates and eigenvalues of this system? One can once again start from a ground state $|0\rangle$ for which

$$c|0\rangle = 0 \tag{12.38}$$

and introduce higher excited states by acting with $c^\dagger$. However, from (12.35) we see that we can only apply $c^\dagger$ once. Thus, this system only has two energy eigenstates:

$$|0\rangle \qquad \text{and} \qquad |1\rangle = c^\dagger|0\rangle \tag{12.39}$$

with energies $E_0 = -1/2$ and $E_1 = +1/2$. One can think of this system as, for example, describing the two spin states of an electron.

An arbitrary state in the Hilbert space would of course be a linear combination of the basis states:

$$|f\rangle = f_0|0\rangle + f_1|1\rangle. \tag{12.40}$$

with $f_0, f_1 \in \mathbb{C}$. This suffices to describe the physical state space of the fermionic harmonic oscillator, but if we want to describe the system in path integral language, it turns out to be useful to introduce the following mathematical trick: we allow f_0 and f_1 to be elements of a *Grassmann algebra* spanned by a single Grassmann variable θ and its complex conjugate θ^*.

In particular, this allows us to introduce a particular state $|\theta\rangle$:

$$|\theta\rangle = |0\rangle + \theta|1\rangle, \qquad \langle\theta| = \langle 0| + \langle 1|\theta^*. \tag{12.41}$$

This state is a so-called *coherent state*: it is an eigenstate of the annihilation operator,

$$c|\theta\rangle = \theta|0\rangle = \theta|\theta\rangle. \tag{12.42}$$

One important use of this state is that we can now write the completeness relation in an interesting form (see exercise 3c):

$$\begin{aligned} I &= |0\rangle\langle 0| + |1\rangle\langle 1| \\ &= \int d\theta^* d\theta\ e^{-\theta^*\theta}|\theta\rangle\langle\theta|. \end{aligned} \tag{12.43}$$

Similarly, we can now compute the trace of the operator $e^{-\beta H}$ and find (see exercise 3d):

$$\mathrm{Tr}\ e^{-\beta H} = \int d\theta^* d\theta\ e^{-\theta^*\theta}\langle-\theta|e^{-\beta H}|\theta\rangle \tag{12.44}$$

We urge the reader to do this computation, and in particular to make sure that the bra vector on the left needs to be $\langle-\theta| = \langle 0| - \langle 1|\theta^*$ for the signs to work out.

The relations (12.44) and (12.43) form the basis of path integral computations like the ones we performed in lecture 2. In fact, in (12.44) we can now write

$$e^{-\beta H} = \prod_{i=1}^{N} e^{-\delta t H} = \prod_{i=1}^{N}(1 - \delta t H + \mathcal{O}(\delta t^2)) \tag{12.45}$$

where $\delta t = \beta/N$, and insert the completeness relation (12.43) between every two factors in the product. The resulting computation proceeds as in lecture 2 (though there are some

subtleties in the particular case of the fermionic harmonic oscillator; see section 1.5 in Nakahara for details) and the end result is that one finds a Lagrangian L for which

$$\text{Tr } e^{-\beta H} = \int_{\text{APBC}} D\theta^*(t)D\theta(t) \exp\left(-\int_{t=0}^{t=\beta} L[\theta(t)]dt\right). \tag{12.46}$$

Here, roughly speaking the Lagrangian is $L = \theta^*\dot{\theta} + \theta^*\theta$, but as mentioned, there are some subtleties involving regularization of infinities (see Nakahara) that we will not go into here.

What matters to us is the fact that from (12.44) we see that the path integral has *anti-periodic* boundary conditions: we can think of the fermionic "paths" $\theta(t)$ as "starting" at θ and "ending" at $-\theta$. Of course, $\theta(t)$ is a Grassmann variable and can not really "take on a value", so all of these words would only have their literal meaning if θ was a bosonic variable; it is however often useful to think about $\theta(t)$ in those terms.

Now, recall that in our index theorem proof, we are *not* interested in computing $\text{Tr } e^{-\beta H}$, but we want to compute

$$\text{ind}(D) = \text{Tr } (-1)^F e^{-\beta H} \tag{12.47}$$

instead. Fortunately, the insertion of the operator $(-1)^F$ is now eaily performed. In our fermionic harmonic oscillator example, we have

$$\begin{aligned}
\text{Tr } e^{-\beta H} &= \langle 0|e^{-\beta H}|0\rangle + \langle 1|e^{-\beta H}|1\rangle \\
\text{Tr } (-1)^F e^{-\beta H} &= \langle 0|e^{-\beta H}|0\rangle - \langle 1|e^{-\beta H}|1\rangle
\end{aligned} \tag{12.48}$$

Thus, all we need to do is make sure that the term involving $|1\rangle$ appears with a relative minus sign. Looking at (12.44) and noticing that

$$\langle \pm\theta| = \langle 0| \pm \langle 1|\theta^* \tag{12.49}$$

we see that we in fact have

$$\text{Tr } (-1)^F e^{-\beta H} = \int d\theta^* d\theta \, e^{-\theta^*\theta} \langle\theta|e^{-\beta H}|\theta\rangle. \tag{12.50}$$

Thus, the insertion of $(-1)^F$ turns the anti-periodic boundary conditions into periodic ones, and we can write

$$\text{Tr } e^{-\beta H} = \int_{\text{PBC}} D\theta^*(t)D\theta(t) \exp\left(-\int_{t=0}^{t=\beta} L[\theta(t)]dt\right). \tag{12.51}$$

This result, which we explicitly derived for the fermionic harmonic oscillator, is in fact a generic one: whenever we want to compute an expression of the form $\text{Tr } e^{-\beta H}$ we obtain a path integral with anti-periodic boundary conditions on the fermions (and periodic ones on the bosons), but insertion of a $(-1)^F$ operator turns the fermionic boundary conditions into periodic ones (and leaves the bosonic boundary conditions unchanged).

So, we now understand why the path integral in our index computation has periodic boundary conditions. We now turn to our final question: what is the Lagrangian we need to use in this path integral?

12.4 A supersymmetric Lagrangian

We would like to find a Lagrangian L such that we can rewrite

$$\text{ind}(D) = \text{Tr } (-1)^F e^{-\beta H} \tag{12.52}$$

as

$$\text{ind}(D) = \int_{\text{PBC}} Dx D\psi \exp\left(-\int_{t=0}^{t=\beta} L[x,\psi]dt\right) \tag{12.53}$$

for a generic ellicptic Fredholm operator D. Of course, the precise form of the Lagrangian and the choice of the fields x and ψ depend on the operator D, but the general strategy is always the same and consists in viewing the operator Q that was introduced in (12.19) as a supersymmetry operator.

In fact, in what follows we will work backwards: we start from specific supersymmetric Lagrangians, and see what Q, D and H are for those. First of all, let us look at the simple example of a massless supersymmetric quantum mechanical particle on $\mathbb{R}^{2n}$. It has $2n$ coordinates $x_i(t)$, as well as $2n$ associated fermionic coordinates $\psi_i(t)$. The Lagrangian for this particle is very simple: it is

$$L = \frac{1}{2}\dot{x}^i\dot{x}_i + \frac{i}{2}\psi^i\dot{\psi}_i \tag{12.54}$$

From this expression, it is straightforward to compute the operator H: one first computes the momenta

$$p_i = \frac{dL}{d\dot{x}^i} = \dot{x}_i, \qquad \pi_i = \frac{dL}{d\dot{\psi}^i} = -\frac{i}{2}\psi_i. \tag{12.55}$$

Then we use the relation between Hamiltonian and Lagrangian:

$$\begin{aligned} H &= \dot{x}^i p_i + \dot{\psi}^i \pi_i - L \\ &= \frac{1}{2}p^i p_i. \end{aligned} \tag{12.56}$$

Note that the fermions drop out. Using the usual coordinate representation $p_i = -i\frac{\partial}{\partial x^i}$, we see that up to a factor, H is the ordinary Laplacian:

$$H = -\frac{1}{2}\Delta. \tag{12.57}$$

However, what is the "square root" operator Q and the operator D that we are computing the index of? To figure this out, we need to study the supersymmetry of the above system, so we can find the map between bosons and fermions that squares to (minus) the Hamiltonian. There is in fact a transformation that turns bosons into fermions:

$$\delta x_i = i\epsilon\psi_i, \qquad \delta\psi_i = -\epsilon\dot{x}_i, \tag{12.58}$$

with ϵ a Grassmann number. In exercise 4, you will check that this transformation is a symmetry: the Lagrangian L changes by a total derivative, and hence the action is invariant. **Noether's theorem** now tells us that Q is the operator that implements this symmetry: we need to find a Q such that

$$\delta x_i = [x_i, \epsilon Q], \qquad \delta \psi_i = [\psi_i, \epsilon Q], \tag{12.59}$$

where we use the usual (anti-) commutation relations

$$[x^i, p_j] = \{\psi^i, \pi_j\} = i\delta^i_j. \tag{12.60}$$

The reader can check (see exercise 4) that the operator that does this is

$$Q = \psi^k p_k. \tag{12.61}$$

What is the coordinate representation of this operator? Clearly, we must use $p_k = -i\partial_k$ again, but what about ψ^k? Note that the second commutator in (12.60) implies that

$$\{\psi^i, \psi_j\} = -2\delta^i_j \tag{12.62}$$

Thus, up to a factor of i, the ψ^i span a Clifford algebra:

$$\psi^i \simeq i\gamma^i. \tag{12.63}$$

Therefore, the operator Q can be written as

$$Q = \psi^i \pi_i \simeq \gamma^i \frac{\partial}{\partial x^i}. \tag{12.64}$$

We see that Q (and therefore, up to a factor of i, D) is nothing but the Dirac operator! We find that with the particular Lagrangian L in (12.54), one computes the index of the Dirac operator on $\mathbb{R}^{2n}$:

$$\text{ind}\left(\gamma^i \frac{\partial}{\partial x^i}\right) = \int_{\text{PBC}} Dx D\psi \exp\left(-\int_{t=0}^{t=\beta} L[x, \psi] dt\right). \tag{12.65}$$

Without computation (see Nakahara, section 12.9.3 for details), we state that the above can straightforwardly be generalized to the Dirac operator on a *curved* even-dimensional manifold: one then uses the covariantization of (12.54):

$$L = \frac{1}{2} g_{\mu\nu}(x) \left(\dot{x}^\mu \dot{x}^\nu + \psi^\mu D_t \psi^\nu\right) \tag{12.66}$$

where D_t is the covariant derivative:

$$D_t \psi^\nu = \dot{\psi}^\nu + \dot{x}^\lambda \Gamma^\nu_{\lambda\kappa}(x) \psi^\kappa, \tag{12.67}$$

with $\Gamma^\nu_{\lambda\kappa}(x)$ the Christoffel symbol for the metric $g_{\mu\nu}$. Using this Lagrangian, one therefore computes the index of the Dirac operator on a curved manifold.